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ERKKI K. LAITINEN

Nonsteady Corporate Model in Bankruptcy Prediction

ABSTRACT

The traditional technique in bankruptcy prediction is to estimate a static cross-sectional multivariate financial ratio model to discriminate between bankrupt and nonbankrupt firms with a minor attention to the dynamics in corporate progress. The purpose of this study is to present a nonsteady corporate model (allowing the growth and profitability of the firm to change over time) and to test whether the parameter estimates of such a model will include incremental information over traditional financial ratios. The discriminatory power of the corporate model parameter estimates is tested in a sample of 42 failed and 42 nonfailed Finnish limited companies. The financial data available cover a time series of seven years prior to bankruptcy.

The parameters of the nonsteady model are estimated by the Marquardt (1963) iterative method. Stepwise logistic regression is applied as the statistical method to discriminate between failed and nonfailed firms. The set of variables used in this logistic analysis consisted of the estimates for the nonsteady model parameters and of the variables used as a benchmark (seven financial ratios and variables), all from the first year before bankruptcy. The results show that the parameter estimates associated with the steadiness of growth (growth model) and the firm-level profitability (profitability model) are significant variables to discriminate between bankrupt and nonbankrupt firms. However, only the growth model parameter estimate includes incremental information over the benchmark variables and improves the classification accuracy of the benchmark model. The parameter estimates referring to periodic (not firm-level) profitability or its changes prove not to have any discriminatory power.

Key words: *Bankruptcy prediction – Corporate model – Growth – Profitability – Logistic Analysis*

ERKKI K. LAITINEN, Professor
Department of Accounting and Business Finance, University of Vaasa • e-mail: ekla@uwasa.fi

1. INTRODUCTION

The traditional technique used in bankruptcy prediction is to estimate a cross-sectional multivariate model, based on financial ratios, to discriminate between bankrupt and nonbankrupt firms (for reviews see Altman 1983, Zavgren 1983, and Jones 1987). Linear discriminant analysis (LDA) was the most popular statistical method for a long time after the innovation by Altman in 1968 (for the critics on LDA see Eisenbeis 1977 and Richardson and Davidson 1984). However, nowadays the most popular multivariate cross-sectional methods are logistic analysis (for the earliest studies see for example Ohlson 1980, Gentry, Newbold, and Whitford 1985, Casey and Bartczak 1985, Zavgren 1985, and Lo 1986) and neural nets (see Tam 1991). The principal idea of these models is to derive a multivariate index (or, a more complex statistical construction) with constant parameters, to be used as a master model to give an approximation of failure risk for a company, on the condition set by the values of the financial ratios in that company.

The main problem in these kinds of cross-sectional models is in their static nature. The models use information only from a given static point of time and are only informative if the failure process remains relatively stable over time. This assumption is usually violated (see for instance Zavgren and Friedman 1988). Several authors have tried to make the model more dynamic by incorporating trends (see Blum 1974, Storey et al 1990, and Laitinen 1993) or stability measures (see Dambolena and Khoury 1980, and Betts and Belhoul 1987) in a static model. Another (and more natural) way to dynamize a static failure model is to use survival analysis (see Lane, Looney, and Wansley 1986, and Luoma and Laitinen 1991). This method uses the survival time as a dependent variable which makes it possible to fit the model for a failure process not following the steady state path. However, survival analysis in its basic form fits the same model to all failed firms, implying that they all behave equally as failure approaches. This assumption is often violated leading to poor results in bankruptcy prediction (for different failure processes see Laitinen 1991).

The most obvious and interesting way to avoid the difficulties above may be the use of a corporate model to predict bankruptcy. This kind of approach allows us to use a firm-specific dynamic model to describe the failure or nonfailure process of firms in a sample. The corporate model approach introduces a set of variables which form a consistent description of the financial process of a company. Thus it remarkably differs from the use of trends and stability measures that are separate and do not form a logical totality. Furthermore, this approach estimates a model of its own for each company, not a uniform master model to be applied to every firm, as does the survival analysis. Suvas (1994) has recently used the corporate model approach to predict failure. He utilized the steady-state (constant ratio) corporate model developed by Ruuhela (1972 and 1975) to project future financial statements (see also Salmi 1982). The model is based on a constant growth and profitability assumptions having much in common with the classic work of Solomon (1966). Suvas used the projected leverage for the sample firms as the classification criterion. His approach outperformed the Altman (1968) Z-model used as a benchmark.

The main shortcoming in the framework applied by Suvas is in the steady-state (constant ratio) nature of the underlying corporate model. The typical failure process is obviously characterized by changes in growth and profitability leading to a deterioration of financial ratios, as is shown by the aforementioned studies. Consequently, to avoid this deficiency, the purpose of the present study is to apply a nonsteady corporate model to predict bankruptcy. This kind of model may provide us with a more realistic framework since the growth and profitability of a firm are allowed to change over time. *The exact objective of the study is to test whether the parameter estimates of such a nonsteady corporate model characterizing growth and profitability will include incremental information over traditional financial ratios in bankruptcy prediction.*

2. NONSTEADY CORPORATE MODEL

The present approach is a nonsteady extension of the Solomon (1966) type of steady corporate model. Solomon assumed that fixed expenditure on investment projects will grow at a constant rate annually and that each annual expenditure generate an identical flow of funds from operations proportional to the size of expenditure (assumption of identical investment projects). Consequently, each annual expenditure has the same profitability measured by the internal rate of return (IRR) equal to the IRR of the whole firm. This classic approach has led to several steady extensions (see for example Sarnat and Levy 1969, Livingstone and Salamon 1970, Ruuhela 1972, and Salamon 1973) and profitability estimation methods (see for instance Brief 1985, Stark 1987, Griner and Stark 1988 and 1991, Kelly and Tippett 1991, and Stark, Thomas and Watson 1992).

The present model will differ from the previous approaches in the expenditure and revenue concepts as well as in its nonsteady nature. The model is based on total (current and fixed) expenditure (instead of fixed expenditure) which is assumed to annually grow at a nonconstant rate and to generate a nonsteady flow of total revenue (net sales plus other revenue) contributions (instead of funds from operations). These concepts in empirical analysis may outperform the previous ones because fixed expenditure and funds from operations in failing companies are usually extremely unstable leading to unreliable estimates for the corporate model.[1] Note that the concept total expenditure does not include interest expenses and taxes because the framework is based on the profitability before these items (as is the return on investment ratio).

The relative form of the revenue contribution flow (income pattern) generated by each annual expenditure is assumed constant as in the Solomon framework, and to follow an infinite geometric distribution. However, the number of revenue units generated by a unit of expenditure, i.e. the monetary productivity, is assumed to change at a constant rate over time. This leads to changes in the IRR. If the monetary productivity is unity, then an expenditure generates an equal amount of revenues and IRR is identically zero. The derivation of the model is in details presented in Laitinen (1997) so that here only the main stages are repeated.

The present approach assumes that each unit of periodic total expendure E_t generates M_t units of revenue which distribute over time according to an infinite geometric distribution with a constant parameter d (for the geometric distribution see Appendix 1). The total number of revenue units generated by E_t is then

$$(1) \qquad N_t = E_t M_t$$

and, followingly, the number of revenue units received in the period t+i is

$$(2) \qquad N_{t+i} = E_t M_t (1-d)\, d^{\,i}$$

according to the geometric revenue distribution.

The IRR of the expenditure spent in the period t is defined as the rate of discount r_t which makes the present value of revenue contributions equal to the expenditure. Thus, for the infinite geometric revenue distribution the IRR of total expenditure spent in the period t can on the basis of (2) implicitly be defined as follows:

$$(3) \qquad E_t = E_t M_t (1-d) \sum_{i=0}^{\infty} d^{\,i} [1+r_t]^{-i} \qquad\qquad d/[1+r_t] < 1$$

where E_t is the expenditure spent in the period t, r_t is the IRR, and M_t the monetary productivity of this expenditure.

The behaviour of r_t in time can be described by introducing the monetary productivity

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[1] The total expenditure and revenue do not usually show as much unsystematic variation in their time series as fixed expenditure and funds from operations do. This will help the empirical applicability of the model. Furthermore, some degree of nonsteadiness (in growth and profitability) is required when the failure process is depicted. The infinity assumption of the contribution flow is necessary to keep the model simple enough.

concept. This productivity is assumed to change at a constant rate over time so that $M_t = M_0(1+h)^t$ where M_0 is the monetary productivity of the expenditure spent in the period 0 and h is the steady rate of change in productivity. Equation (3) allows us to calculate M_t as

$$M_t = M_0\,(1+h)^t = \frac{1+r_t-d}{[1+r_t](1-d)} \tag{4}$$

which makes it possible to solve r_t explicitly as follows:

$$r_t = -1 + \frac{d}{1-M_0\,(1+h)^t\,(1-d)} \tag{5}$$

Equation (5) shows that r_t is in a nonlinear way dependent on the rate of change in M_t.

The total revenue of the firm received in the period t can now be derived on the basis of the previous concepts. This revenue, R_t , is consisted of the revenue contributions generated by the current and past expenditures (E_t , i = t, t–1, t–2,..). R_t can, without specifying the growth function for total expenditure, be presented in the following way for a finite project life:

$$R_t = \sum_{i=t-n}^{t} E_i\, M_i\, \frac{1-d}{1-d^{\,n+1}}\, d^{\,t-i} \tag{6}$$

where n is the project life. Note that (6) has the denominator $1-d^{\,n+1}$ due to the finite geometric distribution (project life) assumed at this stage. Therefore there is not a direct link from (2) to (6). R_t in (6) evidently depends on the shape of the growth function of total expenditure.

R_t can now be specified assuming a special growth path for the total expenditure in (6). Let us assume that the growth function of total expenditure conforms to the second-order Pascal (negative binomial) distribution with the parameter q, i.e. $E_t = E_0\,(1+t)\,q^t$ where q is the growth parameter (for the Pascal distribution see Appendix 1). This distribution can be used to depict both increasing and decreasing rate of growth depending on the values of q. Furthermore, for q greater than unity and for large values of t it can be used to approximate steady state growth. Thus, the second-order Pascal distribution is a very elastic distribution to model a growth function.[2]

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[2] Note that because the growth function of total expenditure, steady change in monetary productivity (steady growth), and the calculation of IRR (discounting) are all based on the family of Pascal distributions (second and first-order) with respect to time, the mathematical formulae can be kept in a reasonable simplicity. The annual growth rate in the period t is $[(1+t)\,q^t/(t\,q^{t-1})]-1 = [q\,(1+t)/t]-1 \approx q-1$ for large t. Thus, for q greater than unity and for large t the second-order Pascal distribution as a growth function can be used to approximate steady state growth.

Let us specify the growth function of E_i as the second-order Pascal distribution as a function of i as described above. Then R_t in (6) is specified as:

$$(7) \qquad R_t = \sum_{i=t-n}^{t} E_0(1+i)\, q^i\, M_0\, (1+h)^i \frac{1-d}{1-d^{\,n+1}}\, d^{\,t-i}$$

$$= \sum_{i=t-n}^{t} E_t\, (1+t)^{-1}\, (1+i)\, q^{-t+i}\, M_t\, (1+h)^{-t+i} \frac{1-d}{1-d^{\,n+1}}\, d^{\,t-i}$$

where q is the parameter of the growth function.

The solution of (7) for a finite n is too complicated to be interpreted and used in further analyses. Thus, let us assume that n may reach infinity and the sum (7) converges. Then the solution of (7) is then (see Appendix 2):

$$(8) \qquad R_t = \frac{E_t\, M_t\, (1+h)(1-d)q\, \{-[(t+2)d-(t+1)q(1+h)]\}}{(1+t)[d-q(1+h)]^2}$$

and using (8) we get the ratio of revenue to expenditure as follows:

$$(9) \qquad \frac{R_t}{E_t} = \frac{[1+r_t-d]\, (1+h)q\, \{-[(t+2)d-(t+1)q(1+h)]\}}{[1+r_t](1+t)[d-q(1+h)]^2}$$

substituting (3) in (6) and simply dividing by E_t. The ratio (9) measures the ratio of the annual revenue to the total expenditure in the period t, i.e. the sufficiency of revenue to finance expenditure. This ratio will be used in empirical analysis to form a restriction to the profitability estimation.

The nonconstant IRR, or r_t , refers to the profitability of the periodic expenditure E_t spent in the period t. However, for the present purpose a firm-level profitability measure would be useful. The firm-level IRR (i.e. weighted sum of the IRRs of all past unexpired expenditures) in that same period can be calculated for the firm with the present nonconstant rate of growth in the following two stages. First, the weighted monetary productivity of unexpired expenditures is needed:

$$(10) \qquad M^f_t = \frac{\sum_{i=0}^{\infty} M_t\, (1+h)^{-i}\, E_t\, (1+t)^{-1}\, (1+t-i)\, q^{-i}\, d^{\,i+1}}{\sum_{i=0}^{\infty} E_t\, (1+t)^{-1}\, (1+t-i)\, q^{-i}\, d^{\,i+1}} = M_t \frac{(1+h)(q-d)^2\, [(1+t)q(1+h)-d(2+t)]}{[q(1+h)-d]^2\, [(1+t)q-d(2+t)]}$$

which shows the number of revenue units to be produced by the expenditure units effective in the the period t.

Second, the firm-level IRR can be obtained by substituting M^f_t expressed in (10) to (5) to yield r^f_t as follows:

$$(11) \quad r^f_t = -1 + \frac{d}{1-M^f_t\,(1-d)}$$

where r^f_t is the IRR for the firm as a whole (for all unexpired expenditures). This measure of firm-level profitability is dependent on the growth of expenditures (q), the distribution of revenue contributions (d), the level of monetary productivity (M_t), and the steady change in monetary productivity (h) as is shown by (10).

3. DATA AND ESTIMATION METHOD

3.1. Empirical data

The data set consists of 42 Finnish bankrupt firms and their nonbankrupt counterparts. The data covers the financial statement information from the bankrupt firms in the period of seven years before bankruptcy. The bankrupt firms failed during the years 1986–1991 and were all over ten years old when bankruptcy occurred. They mainly consisted of smaller firms from several lines of business. However, the majority of the firms were industrial firms. All the firms were limited companies. The nonbankrupt counterparts were selected along two principles: they belonged to the same line of business and were about the same size. The same calendar years were considered for both bankrupt firms and for their nonbankrupt counterparts.

The data are consisted of firms legally declared bankrupt according to Finnish legislation. There are two types of bankruptcies in Finland. First, a firm may, according to the Bankruptcy Act, go bankrupt when it cannot pay its debts when they fall due (liquidity bankruptcy). Second, a firm can, according to the Companies Act, also go bankrupt when the shareholders capital in the balance sheet declines to a certain level, by reason of losses (solidity bankruptcy). This means that the direct cause of bankruptcy may be either poor liquidity and/or poor solidity. Profitability affects the bankruptcy risk only indirectly through its impact on liquidity and solidity. *This must be taken into account when hypothetizing the information content of IRR estimates over liquidity and solidity ratios.*

3.2. Estimation method

The present nonsteady model is nonlinear so that it cannot be estimated with standard statistical methods (for nonlinear estimation methods see Fomby et al 1984, p. 600–619). The estimation results presented in this study are all obtained by the NLIN (NonLINear regression) procedure in the SAS statistical package (see SAS 1989, p. 1134–1194). This procedure produces least-squares or weighted least-squares estimates of the parameters of a nonlinear model. The estimation procedure is here based on the Marquardt (1963) iterative method which, however, may not be the most efficient method (compared for example with the BHHH). This method regresses the residuals onto the partial derivatives of the model with respect to the parameters until the estimates converge. This method is equivalent to performing a series of ridge regressions and is useful when the parameter estimates are highly correlated or the objective function is not well approximated by a quadratic, like in the present models (for the ridge regressions see Fomby et al 1984, p. 300–302). The use of the NLIN procedure allows us to present the model to be estimated in its initial nonlinear form by algebraic mathematics. Thus the following statistical models will also be presented in that form.

The estimation procedure applied here can generally be divided into the estimation of (1) growth and (2) profitability (IRR) of total expenditure. These estimations will be made separately in this order. The parameters to be estimated are the initial value of total expenditute (E_0), growth parameter of the second-order Pascal distribution (q) (growth estimation), the distribution parameter (d), the steady rate of change in monetary productivity (h), and the internal rate of return in the last period (r_t) (profitability estimation). The growth estimation will also include the estimation of the starting period (i.e. point of truncation) for the second-order Pascal distribution, denoted hereafter by t1. The meaning of the parameter will be discussed later.The observed values of variables are denoted by * in the following text.

Because of the very short time-series available (seven observations) some restrictions are used in estimation to reduce the number of estimated parameters[3]*.* While the first period before bankruptcy may be most informative in failure prediction, the restrictions will be based on the last period's observations. First, the growth estimation will be restricted in the way that the growth path must go through the expenditure in the last period (E_t*). This means that only q and t1 are as for estimation. Second, the profitability estimation will be restricted so that the ratio of total revenue to expenditure expressed in (9) holds for the last period's revenue (R_t*) and expenditure (E_t*). This ensures that we use only recent information. Moreover, the parameter d will be directly approximated using the time-series averages of total assets and total

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3 The time-series available for bankruptcy risk evaluation in practice are usually rather short. Furthermore, the failure process does not usually take many years so that longer time-series may not be informative from the perspective of failure prediction. Thus one particular objective was to use estimation methods based on short time series, although the estimates may to some degree suffer from ad hoc features. The ad hoc features were resulted in large asymptotic standard errors of the estimates. However, this was expected because of the small number of observations and the focus was set on the predictive ability, not on the accuracy, of the estimates. Compare the situation with the use of traditional profitability ratios in failure prediction. These ratios are not very accurate measures of profitability due to the arbitrariness of asset valuation and depreciations. Nevertheless, they are used in failure prediction for the sake of their predictive power.

expenditures for each firm. Therefore, only h has to be estimated by the nonlinear method in profitability estimation.

Growth estimation

The nonsteady growth function is described in (7) by the second-order Pascal distribution. This distribution is very flexible provided that we can freely choose the time interval of the function (i.e. the starting or truncation point) which will be applied to depict the growth of total expenditure. This can be done introducing a parameter of truncation t1 so that:

$$(12) \quad E_i = E_o\,(1+t1+i)\,q^{t1+i} \qquad i = 1,2,\ldots,t$$

where t1 is the starting period for the interval over which our analysis applies. The function in (12) may approximate steady state growth when t1 is large and q larger than unity. Similarly, when t1 is small, this function may, for certain values of q, describe a growth path with a peak. Thus the truncation point t1 helps us to describe alternative growth paths for identical values of q. The higher t1, the closer the growth is to a steady path. *Therefore, the truncation parameter is a measure of the steadiness of the growth which may be useful in failure prediction.*

Let us use, in growth estimation, a restriction based on the last observed value of total expenditure or E*(t). Let us assume that:

$$(13) \quad E_0 = E_t^*\,(1+t1+t)^{-1}\,q^{-t1-t}$$

and so we get:

$$(14) \quad E_i = E_t^*\,\frac{(1+\mathbf{t1}+i)}{(1+\mathbf{t1}+t)}\,\mathbf{q}^{i-t} \qquad i = 1,2,\ldots,t$$

which is the equation that will be used in estimation. This equation includes t1 and q as for estimation (presented in bold). Statistically, equation (14) means to estimate t1 and q using the actual time-series of total expenditure E_i^* (i = 1,2, .., t) as the dependent variable and the time index i (i = 1,2, .., t) as the independent variable in the nonlinear model. The value of the parameter t1 were in estimation constrained to be below 20[4].

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[4] The upper constraint of 20 for t1 was considered as large enough to allow the approximation of a steady growth process. For example, when q = 1.10 the annual growth rate for t = 20 is according to the second-order Pascal distribution (20/19)1.10–1= 0.158 and for t=21 respectively (21/20)1.10 = 1.55. These annual rates are rather close to each other providing us with a reasonable approximation of a steady rate.

Profitability estimation

Let us first consider profitability estimation under nonsteady conditions without any restriction for the growth. For the estimation purposes we need the Koyck transformation which tells that in the geometric lag model the revenue R_i can additively be separated into two parts, the contribution of past expenditures ($R_i d$) and the contribution of the recent expenditure ($E_i M_i (1–d)$, see (2)).[5] The Koyck transformation applied to the geometric lag model in (7) leads to the following equation used in estimation:

$$(15)\quad R_i = R_{i-1}\, d + E_i\, M_0\, (1+h)^i\, (1-d)$$

$$= R_{i-1}\, \mathbf{d} + E_i \frac{1+\mathbf{r}_o - \mathbf{d}}{1+\mathbf{r}_o} (1+\mathbf{h})^i \qquad i = 1,2, \ldots , t$$

which is based on the assumption that monetary productivity M_i is changing at the steady rate h over time i (i = 1,2, .. , t). Equation (15) includes the parameters r_0, d, and h (in bold) to be estimated from a time-series model in which R_i*(dependent variable) is explained by R_{i-1}* and E_i*(independent variables). These estimates can be used to yield r_i through (5).

The number of the parameters (3) in (15) is obviously too large for the present number of observations (7). Therefore, the lag parameter of the revenue distribution, d, will be directly approximated in the following simple way. The average of the geometric revenue distribution (the average time lag between expenditure and generated revenue flow) is defined as d/(1–d). This average time lag is approximated for each firm by the weighted average of the ratio of total assets to total expenditure from the seven years observation period. The ratio calculated in this way roughly tells us how many years total expenditure on an average is hold as assets in the balance sheet before it is expired when generated revenues are accumulated. Let us denote this weighted average as A. Then d is approximated by d = A/(1+A). This approximation is used in the following estimation.

Furthermore, let us add a restriction for the value of r_0 in the form of M_0 presented in (9), to reduce the number of the parameters as for estimation to one.The use of the restriction is based on R_t* and E_t* as well as on the assumption that the growth path follows the second-order Pascal distribution. The restriction based on (9) is as follows:

$$(16)\quad M_0 = \frac{R_t^*(1+t1^*+t)[d^*-q^*(1+h)]^2}{E_t^*(1+h0)^{t1^*+t+1}\,(1-d^*)q^*\,\{-[(t1^*+t+2)d^*-(t1^*+t+1)q^*(1+h)]\}}$$

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[5] The Koyck transformation is used to transform the geometric lag model to the autoregressive form as (13). See for example Fomby et al 1984, p. 382.

where t1* and q* are the parameter values of t1 and q estimated in the earlier phase (growth estimation) and d* the value of d approximated as explained above. Note that there is only one unknown parameter (i.e. h) in Equation (16).

Substituting the restriction (16) in (15) we get the equation to be used in estimation as:

$$(17)\ R_i = R_{i-1}\, d^* + \frac{E_i\,(1+\mathbf{h})^{i-t-1}\, R_t^*\,(1+t1^*+t)[d^*-q^*(1+\mathbf{h})]^2}{E_t^*\, q\, ^*\{-[(t1^*+t+2)d^*-(t1^*+t+1)q^*(1+\mathbf{h})]\}} \qquad i = 1,2,\ \ldots\ ,t$$

Equation (17) includes only the parameter h (in bold) as for estimation from a time-series model in which R_t^* is explained by R_{i-1}^* and E_i^*. The value of h was in estimation constrained to be $-0.02 \leq h \leq 0.02$.[6] The estimate of the last period's IRR or r_t can be calculated by substituting the approximated d and the estimate of h in (16) and (5).

3.3. Variables and failure prediction method

The basic concepts of the present growth and profitability models are total revenue and total expenditure. Total revenue is measured by the sum of net sales and other revenue. Total expenditure is calculated as the sum of current expenditure, fixed expenditure, taxes and interest expenditure. The present empirical analysis consists of several phases. First, the nonsteady models based on the basic concepts are estimated from the time-series of these variables. Second, the parameters of the models are then used as variables in a stepwise logistic bankruptcy prediction model. Third, a stepwise logistic model based on ordinary financial variables is estimated to form a benchmark model. Fourth, the incremental nature of the information content of the nonsteady model variables over the ordinary financial variables is evaluated estimating a combined logistic model from data consisted of both data sets.

Panel A in Table 1 presents the eight nonsteady model variables used in the first phase logistic analysis. The set of variables consists mainly of model parameter estimates but also of variables based on these estimates. These variables also include two predictions for the year t+1 (bankruptcy year) calculated with the aid of the estimated corporate model. The variables 3 (g_{t+1}) and 7 (r_{t+1}) are the next year predicted values for the growth rate and for the IRR, respectively. Note that the variable 6 (r_t) is the estimate for the IRR of the last period's expenditure while the variable 8 (r^f_t) refers to IRR at the level of the firm (firm-level profitability).

Panel B in the same table shows the list of the seven ordinary financial variables used in the logistic benchmark model. The set of variables is made up of four financial ratios and of

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[6] This was made because of the sensitivity of r_t to h. For example, if d=0.5, M_0=1.05, and h=+0.02, then according to (3) r_0 =+0.053 and r_{10}=+0.389. Similarly, if d=0.5, M_0=1.05, and h=–0.02, then r_0 =+0.053 and r_{10} = –0.124. These figures show that the range for h (–0.02,+0.02) is large enough to allow the profitability to dramatically change under a ten year's period.

three other variables. The financial ratios are chosen to measure the main financial dimensions, i.e. profitability, liquidity (in dynamic and static perspective) and solidity. The set also includes a variable for short-term growth and size. Finally, there is a variable for the revenue-expenditure relationship. This variable is included to ensure that possible incremental information yielded by the nonsteady model variables does not originate from the restriction set on that relationship in estimation.

All the statistical calculations associated with bankruptcy prediction are carried out by means of the SAS statistical package. The logistic models are estimated by the LOGISTIC procedure in the stepwise model (see SAS 1989 p. 1071–1126 and for the logistic regression for example Hosmer and Lemeshow 1989). The goodness of fit in the logistic model is evaluated by the Schwartz Criterion. This statistic gives a way of adjusting the $-2\mathrm{Log}L$ statistic for the number of variables[7]. Because of the obvious nonnormality of the variable distributions the univariate differences between the bankrupt and nonbankrupt firms are tested by a nonparametric statistic using the NONPAR1WAY procedure (see SAS 1989 p. 1195–1210). This was done by testing the similarity of the location of the distributions of the variables in the both groups by the Kruskal-Wallis test based on Wilcoxon scores[8]. The significance (probability) level used in analysis for the test is 5 % (with one degree of freedom).

3.4. Hypotheses

The empirical results are obviously affected by the fact that the benchmark model is estimated from the financial data in the first year prior to failure. Laitinen (1993) has shown that in the first year before bankruptcy, the levels of financial ratios include so much information (i.e. they are so poor for failing firms) that trends or differences of the ratios may not bring any incremental information in bankruptcy prediction. *Thus it can be hypothetized that the estimate of the parameter* h*, percentage steady change in monetary productivity, does not include incremental information over* r_t *, periodic profitability, or* r^f_t *, firm-level profitability in bankruptcy prediction.* The logical expectation is that in earlier stages of bankruptcy this hypothesis should be altered. However, the present data does not allow to test that kind of hypothesis.

The firm-level profitability is expected to be a more important predictor of failure than the periodic one, since, as late as in the first year prior to bankruptcy, the economic state of the whole firm is decisive. This firm-level profitability may include much the same information as the return on investment ratio (ROI) in the benchmark model. However, bankruptcy in

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[7] The $-2\mathrm{Log}L$ statistic has a chi-square distribution under the null hypothesis that all the esplanatory variables in the model are zero. Schwartz Criterion (SC) is defined as $SC = -2\ \mathrm{Log}L + (k+s)\log(N)$ where k is the number of ordered values for the response, s the number of explatory variables and N the total number of observations.

[8] Wilcoxon scores are the ranks a(Rj) = Rj where Rj the rank of the jth observation and a(Rj) is the rank score. These ranks are locally most powerful for location shift of a logistic distribution.

TABLE 1. List of variables used in logistic models

PANEL A. NONSTEADY CORPORATE MODEL VARIABLES

GROWTH:
1. **q**
2. **$t1$**
3. **predicted growth rate for $t+1 =: g_{t+1}$**

PROFITABILITY:

4. **d**
5. **h**
6. **r_t**
7. **predicted internal rate of return for $t+1 = r_{t+1}$**
8. **r^f_t**

PANEL B. FINANCIAL BENCHMARK MODEL VARIABLES

PROFITABILITY:
1. **Return on investment ratio =100*Net profit/Total assets (ROI)**

DYNAMIC LIQUIDITY:

2. **Traditional cash flow = 100*Traditional cash flow/Net sales (CFW)**

STATIC LIQUIDITY:

3. **Quick ratio = Financial assets/Current debt (QCK)**

SOLIDITY:

4. **Shareholders capital to total assets ratio = 100*Shareholders capital/Total assets (SCA)**

SHORT-TERM GROWTH:

5. **Growth rate in net sales = 100*(Net sales$_t$ – Net sales$_{t-1}$)/Net sales$_{t-1}$ (GRO)**

SIZE:

6. **Total revenue (REV)**

REVENUE-EXPENDITURE RELATIONSHIP:

7. **Revenue-expenditure ratio = Total revenue/Total expenditure (TRE)**

Finland is a direct consequence of poor liquidity (liquidity bankruptcy) or poor solidity (solidity bankruptcy). *This means that, in the first year before failure, it is expectable that profitability measures are not as good predictors as solidity or liquidity ratios because profitability only indirectly (in a lagged way through liquidity and solidity) affects the bankruptcy risk.* [9] When

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[9] Note that the situation could be reversed when earlier data were analysed because, say, two or three years before bankruptcy, poor profitability may not yet have affected liquidity and solidity as strongly as in the first year prior to failure. Consequently, in these early years poor profitability measures may include predictive power as much as or more than liquidity and solidity ratios which may yet be at a moderate level.

earlier years were evaluated, the importance of profitability would be much larger. This is because often poor profitability level and a decline in this level act as the trigger for the start of the failure process already years before bankruptcy. Thus, the present study only yields preliminary results due to the nature of the data. In the future, the applicability of this kind of framework should be tested using longer time series.

Similarly, the growth of the firm has obviously no direct effect on bankruptcy risk while having a lagged and indirect effect through solidity and liquidity. However, it is expectable that the present long-term nonsteady growth model may include some incremental information over the periodic growth rate (GRO) and other financial variables in the benchmark model. Let us imagine two firms with identical financial ratios and an identical positive yearly growth rate. The growth of the former firm has followed a steady path while the latter firm's growth rate has been nonsteady and decreasing. While having identical financial ratios, the economic situation in the former (steady) firm may be much better under management control and is, consequently, associated with a smaller bankruptcy risk. *Thus, it is expected that the parameter estimates of the growth model have incremental information over the benchmark model variables.*

4. EMPIRICAL RESULTS

Table 2 presents the quartiles of the model variables (Panel A) and the benchmark variables (Panel B) in both bankrupt and nonbankrupt firms. Panel A shows that all the three growth model variables differ significantly at the 5 % risk level. The truncation parameter (t1) referring to steadiness of growth is the best univariate discriminator (the higher t1, the steadier is the growth rate). However, at this risk level none of the profitability model variables differs significantly between bankrupt and nonbankrupt firms. The most prominent differences are found in the firm-level and periodic internal rate of return (r^f_t and r_t). The firm-level IRR slightly ourperforms the periodic IRR as a univariate discriminator. The differences in the financial benchmark variables are all statistically significant except those in REV (total revenue) and TRE (total revenue to total expenditure ratio)[10].

The stepwise estimated logistic model based on the model variables is presented in Table 3. The resultant model only includes (in addition to the intercept) the variables t1 (truncation variable in the growth model) and r^f_t (firm-level IRR in the profitability model). Hence also in a multivariate analysis r^f_t outperforms r_t , which supports our previous expectations.Table

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[10] It was expected that there are no statistically significant differences in the size between failed and nonfailed firms due to the pairwise sampling.

TABLE 2. Quartiles of the variables

	Bankrupt firms			Nonbankrupt firms			
	25 %	50 %	75 %	25 %	50 %	75 %	K-W
PANEL A. NONSTEADY CORPORATE MODEL VARIABLES							
Growth model:							
1. q	0.86	0.92	0.99	0.91	1.02	1.06	0.0331
2. t1	1.00	1.00	3.41	1.00	6.81	20.00	0.0076
3. g_{t+1}	–4.99	–0.04	8.37	–1.10	6.82	13.73	0.0513
Profitability model:							
4. d	0.31	0.36	0.43	0.31	0.35	0.42	0.8555
5. h	–0.02	–0.01	0.02	–0.02	–0.00	0.02	0.4611
6. r_t	–0.15	–0.04	0.15	–0.06	0.05	0.21	0.0619
7. r_{t+1}	–0.17	–0.05	0.18	–0.11	0.04	0.17	0.2378
8. r^f_t	–0.14	–0.03	0.14	–0.05	0.06	0.18	0.0530
PANEL B. FINANCIAL BENCHMARK MODEL VARIABLES							
1. ROI	–5.14	2.14	8.88	4.26	8.99	14.44	0.0001
2. CFW	–11.94	–3.72	0.06	–1.27	1.59	7.22	0.0001
3. QCK	0.29	0.40	0.58	0.54	0.76	1.05	0.0005
4. SCA	2.08	4.06	11.54	7.26	16.33	33.05	0.0001
5. GRO	–20.84	1.72	14.27	–6.34	11.90	29.30	0.0413
6. REV	5993	11979	27823	5578	14052	34360	0.6100
7. TRE	0.86	0.95	1.10	0.94	1.00	1.06	0.4388

Legend: For variables see Table 1.
K-W = The significance level of the Kruskal-Wallis non-parametric test statistic based on Wilcoxon scores on the similarity of the location of the distributions

4 presents the resultant logistic model for the benchmark variables. This model also consists of two variables, the traditional cash flow (CFW) and the shareholders capital to total assets ratio (SCA). The Schwartz Criterion shows that this model significantly ourperforms the previous model (based on the model variables) in goodness of fit. Thus, the ordinary financial ratios used as a benchmark evidently lead to a better discrimination between bankrupt and non-bankrupt firms than the model variables in the first year prior to bankruptcy. *This was expected since the model variables do not contain the liquidity and solidity aspects which are direct causes of bankruptcy.*

TABLE 3. Logistic model based on the model variables

PANEL A. STEPWISE LOGISTIC MODEL

Variable	Estimate	Standard error	Chi-Square	Probability
Intercept	0.8400	0.3445	5.9422	0.0148
t1	–0.0890	0.0296	9.0058	0.0027
r^f_t	–3.8695	1.6041	5.8191	0.0159

Schwartz Criterion 114.067

PANEL B.VARIABLES NOT IN THE MODEL

	Score Chi-Square	Probability
q	1.4078	0.2354
g_{t+1}	1.2869	0.2566
d	0.0008	0.9769
h	2.8442	0.0917
r_t	1.1157	0.2909
r_{t+1}	2.3574	0.1247

TABLE 4. Logistic model based on the benchmark variables

PANEL A. STEPWISE LOGISTIC MODEL

Variable	Estimate	Standard error	Chi-Square	Probability
Intercept	0.5554	0.4132	1.8062	0.1790
CFW	–0.2510	0.0715	12.3393	0.0004
SCA	–0.0698	0.0276	6.3815	0.0115

Schwartz Criterion 84.171

PANEL B.VARIABLES NOT IN THE MODEL

	Score Chi-Square	Probability
ROI	0.0251	0.8741
QCK	0.1614	0.6879
GRO	1.2385	0.2658
REV	1.1388	0.2859
TRE	0.5504	0.4581

Table 5 presents the combined logistic model which is based on three variables, t1 (growth model), CFW and SCA (benchmark model). Thus the growth model seems to include some incremental information over the benchmark variables while the profitability model does not.

TABLE 5. Logistic model based on the model and benchmark variables

PANEL A. STEPWISE LOGISTIC MODEL

Variable	Estimate	Standard error	Chi-Square	Probability
Intercept	1.2475	0.5419	5.2984	0.0213
t1	–0.0749	0.0348	4.6317	0.0314
CFW	–0.2424	0.0737	10.8165	0.0010
SCA	–0.0267	0.0267	7.1867	0.0073

Schwartz Criterion 83.677

PANEL B.VARIABLES NOT IN THE MODEL

	Score Chi-Square	Probability
q	0.5721	0.4494
g_{t+1}	0.6430	0.4226
d	0.1056	0.7452
h	0.3385	0.5607
r_t	2.2770	0.1313
r_{t+1}	0.7659	0.3815
r^f_t	2.6064	0.1064
ROI	0.1766	0.6743
QCK	0.2038	0.6516
GRO	0.0726	0.7876
REV	1.4191	0.2336
TRE	1.6142	0.2039

TABLE 6. Cross-validated classification results for the three logistic models

	Type I Error, %	Type II Error, %	Overall Error, %
1. Nonsteady model variables model	50.0	19.0	34.5
2. Benchmark model	16.7	28.6	22.6
3. Combined model	16.7	19.0	17.9

The Schwartz Criterion shows that the contribution in the goodness of fit due to truncation parameter, t1, is not very large. However, this variable significantly improves the classification accuracy of the benchmark model which is shown by the cross-validated (Lachenbruch leaving-one out) classification results in Table 6. Especially, the Type II Error percentage is diminished. The truncation parameter measures the steadiness of the growth in total expenditure.

Summarizing, liquidity (CFW), solidity (SCA), and steadiness of growth (t1) seem to be the main factors including predictive power in failure prediction[11].

Table 5 shows that, on the basis of Score Chi-Square, r^f_t has the highest incremental discriminating power of the variables not in the logistic model. This firm-level IRR certainly outperforms the familiar profitability measure, ROI, as a incremental discriminator. However, an additional experiment performed, showed that the cross-validated classification results were somewhat impaired when r^f_t was taken within the logistic model.

5. SUMMARY OF THE STUDY

The purpose of the present study was to present a nonsteady corporate model (allowing the growth and profitability of the firm to change over time) and to test whether the parameter estimates of such a model will include incremental information over traditional financial ratios. The model was based, firstly, on the assumption that the growth path of the modelled firm follows the second-order Pascal distribution which allows the growth rate to vary over time. Secondly, it was assumed that the internal rate rate of return of periodic investment projects may also vary over time depending on the steady rate of change in monetary productivity (the ratio of revenue units to investment expenditure). This model may have superiour features in comparison to traditional steady models in depicting the unsteady behaviour of failing firms.

The discriminatory power of the model was tested in a sample of 42 failed and 42 nonfailed Finnish limited companies. The financial data available covered a time-series of seven years prior to bankruptcy. Because of the short-time series the model was only estimated for the first year before bankruptcy. Seven financial ratios and other variables were used to form a benchmark model. Stepwise logistic regression was used as the statistical method. The results showed that the parameters associated with the steadiness of growth (growth model) and firm-level profitability (profitability model) are significant variables to discriminate between bankrupt and nonbankrupt firms. *However, only the growth model parameter estimate included incremental information over the benchmark variables and improved classification accuracy.*

The empirical results may be affected by the fact that only the first year before failure was considered. The use of earlier data could lead the parameter estimates associated with the periodic profitability and its change, to outperform the firm-level profitability and benchmark variables as discriminators. The first year before failure for bankruptcy firms is characterized by very poor levels of financial ratios. Especially the ratios of liquidity and solidity (the direct

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[11] These results are consisted for example with Dambolena and Khoury (1980).

causes of failure) usually include a lot of discriminatory power. Followingly, profitability (an indirect cause) estimates are often ourperfomed by these ratios in that year. However, a firm with poor liquidity and solidity ratios may have a smaller bankruptcy risk under a steady growth than under a nonsteady growth. This line of thinking was supported by the incremental information included in the truncation parameter estimate (a measure of the steadiness of the growth).

The results of the present study are preliminary. However, they show that a corporate model approach may prove as an effective method in failure prediction. There are three interesting trends available for further study. First, the analysis should use a longer time-series and also years other than the first year before bankruptcy. Second, efforts should be directed to developing models also incorporating solidity and liquidity in the framework of profitability and growth (for such a model see Suvas 1994). Third, there is also need for developing and testing new statistical methods to estimate the parameters of nonsteady corporate models.

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REFERENCES

ALTMAN, EDWARD I. (1968): Financial Ratios, Discriminant Analysis, and the Prediction of Corporate Bankruptcy. The Journal of Finance. September. 4. 589–609.

ALTMAN, EDWARD I. (1983): Corporate Financial Distress. A Complete Guide to Predicting, Avoiding, and Dealing with Bankruptcy. John Wiley & Sons, Inc.The United States of America.

BETTS, J. – BELHOUL, D. (1987): The Effectiveness of Incorporating Stability Measures in Company Failure Models. Journal of Business Finance & Accounting. Autumn. 323–334.

BLUM, M. (1974): Failing Company Discriminant Analysis. Journal of Accounting Research. Spring. 1–25.

BRIEF, R.P. (1985): Limitations of Using the Cash Recovery Rate to Estimate the IRR: A Note. Journal of Business Finance and Accounting. Autumn. 473–475.

CASEY, CORNELIUS – BARTCZAK, NORMAN (1985): Using Operating Cash Flow Data to Predict Financial Distress: Some Extensions. Journal of Accounting Research. Spring. 384–401.

DAMBOLENA, I.G. – KHOURY, S.J. (1980): Ratios Stability and Corporate Failure. Journal of Finance. 25. 4. 1017–1026.

EISENBEIS, R.A. (1977): Pitfalls in the Application of Discriminant Analysis in Business, Finance and Economics. The Journal of Finance. 32. 875–900.

FELLER, W. (1968): An Introduction to probability and its Applications. Volume I. John Wiley & Sons. New York.

FOMBY, T.B. – HILL, R. CARTER-JOHNSON, STANLEY R. (1984): Advanced Econometric Methods. Springer-Verlag. United States of America.

GENTRY, JAMES A. – NEWBOLD, PAUL – WHITFORD, DAVID T. (1987): Funds Flow Components, Financial Ratios, and Bankruptcy. Journal of Business Finance & Accounting.Winter. 595–606.

GRINER, E.H. – STARK, A.W. (1988): Cash Recovery Rates, Accounting Rates of Return and the Estimation of Economic Performance. Journal of Accounting and Public Policy. Winter. 293–311.

GRINER, E.H. – STARK, A.W. (1991): On the Properties of Measurement Error in Cash Recovery Rate Based Estimates of Economic Performance. Journal of Accounting and Public Policy. Autumn. 207–224.

HALD, A. (1962): Statistical Theory with Engineering Applications. John Wiley & Sons. New York.

HOSMER, D.W. – LEMESHOW, S. (1989): Applied Logistic Regression. John Wiley & Sons, Inc. New York.

JONES, F.L. (1987): Current Techniques in Bankruptcy Prediction. Journal of Accounting Literature. 6. 1. 131–164.

KELLY, G. –TIPPETT, M. (1991): Economic and Accounting rates of Return: A Statistical Model. Accounting and Business Research. Autumn. 321–329.

LAITINEN, E.K. (1991): Financial ratios and Different Failure Processes. Journal of Business Finance and Accounting. 18. 5. 649–673.

LAITINEN, E.K. (1993): Financial Predictors for Different Phases of the Failure Process. Omega. 21. 2. 215–228.

LAITINEN, E.K. (1997): Estimation of Internal Rate of Return Under Nonsteady Conditions. Journal of Business Finance and Accounting. 24(9). October&December. 1217–1251.

LANE, W.R. – LOONEY, S.W. – WANSLEY, J.W. (1986): An Application of the Cox Proportionate Hazards Model to Bank Failure. Journal of Banking and Finance. 11. 511–531.

LIVINGSTONE, J. – SALAMON, G. (1970): Relationship between the Accounting and the Internal Rate of Return Measures: A Synthesis and Analysis. Journal of Accounting Research. Autumn. 199–216.

LO, A. (1986): Logit versus Discriminant Analysis: A Specification Test and Application to Corporate Bankruptcies. Journal of Econometrics. 31. 151–178.

LUOMA, M. – LAITINEN, E.K. (1991): Survival Analysis as a Tool for Company Failure Prediction. Omega. 6. 673–678.

MARQUARDT, D.W. (1963): An Algorithm for Least-Squares Estimation of Nonlinear Parameters. Journal for the Society of Industrial and Applied Mathematics. 11. 431–441.

OHLSON, J.A. (1980): Financial Ratios and the Probabilistic Prediction of Bankruptcy. Journal of Accounting Research. 18. 1. 109–131.

RICHARDSON, F.M. – DAVIDSON, L.F. (1984): On Linear Discrimination with Accounting Ratios. Journal of Business Finance and Accounting. 11. 511–525.

RUUHELA, R. (1972): A Capital Investment Model of the Growth and Profitability of the Firm (in Finnish with Englísh summary). Acta Academiae Oeconomiae Helsiengiensis. Series A:8. Helsinki.

RUUHELA, R. (1975). The Growth and Profitability of the Firm (in Finnish). Oy Gaudeamus Ab. Helsinki.

SALAMON, G. (1973): Models of the Relationship between the Accounting and Internal Rate of Return: An Examination of Methodology. Journal of Accounting Research. Autumn. 296–303.

SALMI, T. (1982): Estimating the Internal Rate of Return from Published Financial Statements. Journal of Business Finance and Accounting. Spring. 63–74.

SARNAT, M. – LEVY, H. (1969): The Relationship of Rules of Thumb to the Internal Rate of Return: A Restatement and Generalization. Journal of Finance. June. 479–490.

SAS (1989): SAS/STAT. User's Guide. Version 6. Fourh Edition. Volume 2. Cary, NC: SAS Institute Inc.

SOLOMON, E. (1966): Return on Investment: The Relation of Book-Yield to True Yield, in Jaedicke, R. – Ijiri, Y. – Nielsen, O.: Research in Accounting Measurement. Menasha, Wisconsin. American Accounting Association. 232–244.

STARK, A.W. (1987): On the Observability of the Cash Recovery Rate. Journal of Business, Finance and Accounting. Spring. 99–108.

STARK, A.W. – THOMAS, H.M. – WATSON, I.D. (1992): On the Practical Importance of Systematic Error in Conditional IRRs. Journal of Business Finance and Accounting. April. 407–424.

STOREY, D. – KEASEY, K. – WATSON, R. – WYNARCZYK, P. (1990): The Performance of Small Firms. Profits, Jobs and Failures. Routledge. 1990. Great Britain.

SUVAS, A (1994): Profitability, Growth and the Prediction of Corporate Failure. The Finnish Journal of Business Economics. 4. 449–468.

TAM, K. (1991): Neural Network Models and the Prediction of Bank Bankruptcy. Omega. 19. 5. 429–445.

ZAVGREN, CHRISTINE V. (1983): The Prediction of Corporate Failure: The State of the Art. Journal of Accounting Literature. 1. 1–38.

ZAVGREN, CHRISTINE V. (1985): Assessing the Vulnerability to Failure of American Industrial Firms: A Logistic Analysis. Journal of Business Finance & Accounting. Spring. 19–45.

ZAVGREN, CHRISTINE V. – FRIEDMAN, GEORGE E. (1988): Are Bankruptcy Prediction Models Worthwhile? An Application in Securities Analysis. Management International Review. 1. 34–44.

APPENDICES

APPENDIX 1. The Pascal distribution

The family of the Pascal distributions (negative binomial distributions) is defined in the following way (see Feller 1968, p. 164–166 and Hald 1962, p. 38–40):

The probability that the v th success occurs at the trial v+k is

$$f(k:v,p) = \binom{v+k-1}{k} p^v q^k = \binom{-v}{k} p^v (-q)^k, \; k = 0,1,2, .. \text{ and } p = 1-q$$

If the parameter v determining the order of the distribution is equal to unity, we have the first-order Pascal distribution as follows:

$$f(k:1,p) = p\, q^k = (1-q)\, q^k, \; k = 0,1,2,...$$

which is identical with the geometric distribution.

If the parameter v is equal to 2, we have the second-order Pascal distribution as follows:

$$f(k:2,p) = (k+1)\, p^2 q^k = (k+1)(1-q)^2 q^k, \; k = 0,1,2,...$$

The first-order Pascal distribution (the geometric distribution) is a monotonically decreasing function of k provided that q is less than unity. This distribution is very popular in depicting growth at a steady rate. The second-order Pascal distribution is also a monotonically decreasing function of k but only when q is less than 0.5. If q exceeds 0.5, then the function has a peak to the right of the origin.

APPENDIX 2. The proof of Equation (8)

Equation (7) states that

$$(7) \qquad R_t = \sum_{i=t-n}^{t} E_t\,(1+i)\,q^i\,M_0\,(1+h)^i\,\frac{1-d}{1-d^{\,n+1}}\,d^{\,t-i}$$

$$= \sum_{i=t-n}^{t} E_t\,(1+t)^{-1}\,(1+i)\,q^{-t+i}\,M_t\,(1+h)^{-t+i}\,\frac{1-d}{1-d^{\,n+1}}\,d^{\,t-i}$$

This equals to

$$(A1) \qquad R_t = E_0\,M_0\,\frac{1-d}{1-d^{\,n+1}}\,d^{\,t-i}\sum_{i=t-n}^{t}(1+i)\,[q(1+h)/d]^{\,i}$$

Let us substitute z = q(1+h)/d in (A1). This leads to

$$(A2) \qquad R_t = E_0\,M_0\,\frac{1-d}{1-d^{\,n+1}}\,d^{\,t}\,[-z^{t+1}[(t+2)-z(t+1)+z^{-n-1}\{(t-n+1)-(t-n)z\}]](1-z)^{-2}$$

Let us assume that z > 1 so that (A2) converges. Let n approach the infinity in (A2) which gives

$$(A3)\ R_t = E_0\,M_0\,(1-d)\,d^t\,\{-z^{t+1}[(t+2)-z(t+1)]\}(1-z)^{-2}$$

and substituting z = q(1+h)/d in (A3) we get

$$(A4) \qquad R_t = E_t\,(1+t)^{-1}\,M_t\,[(1-d)/d]\,(1+h)^{t+1}q\,\frac{\{-[(t+2)d-(t+1)q(1+h)\}d^2}{d\,[d-q(1+h)]^2}$$

$$(A4) \qquad R_t = E_t\,M_0\,(1+h)^{t+1}(1-d)\,q\,\frac{\{-[(t+2)d-(t+1)q(1+h)\}}{(1+t)[d-q(1+h)]^2}$$

which equals (8).